2984. [2004 : 431, 433] *Proposed by Mihály Bencze, Brasov, Romania.*

Prove that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij(i+j)} = 2\sum_{n=1}^{\infty}\frac{1}{n^3}.$$

II. *Composite of essentially the same solutions by Arkady Alt, San Jose, CA, USA; Chip Curtis, Missouri Southern State University, Joplin, MO, USA; Henry Ricardo, Medgar Evers College (CUNY), Brooklyn, NY, USA; and Li Zhou, Polk Community College, Winter Haven, FL, USA.*

Let S denote the given double summation. Then

$$\begin{aligned} S &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{ij}\int_0^1 x^{i+j-1}\,dx = \int_0^1 \left(\frac{1}{x}\left(\sum_{i=1}^{\infty}\frac{x^i}{i}\right)\left(\sum_{j=1}^{\infty}\frac{x^j}{j}\right)\right)dx \\ &= \int_0^1 \frac{\ln^2(1-x)}{x}\,dx. \qquad (1) \end{aligned}$$

Changing variable via $t=-\ln(1-x)$, we have $x=1-e^{-t}$ and $dx=e^{-t}dt$; whence,

$$\begin{aligned} \int_0^1 \frac{\ln^2(1-x)}{x}\,dx &= \int_0^{\infty}\frac{t^2e^{-t}}{1-e^{-t}}\,dt = \int_0^{\infty} t^2\left(\sum_{n=1}^{\infty}e^{-nt}\,dt\right) \\ &= \sum_{n=1}^{\infty}\int_0^{\infty} t^2e^{-nt}\,dt. \qquad (2) \end{aligned}$$

Applying the usual integration by parts twice, we find, after some routine computations involving improper integrals, that

$$\int_0^{\infty} t^2e^{-nt}\,dt = \frac{2}{n^3}. \qquad (3)$$

The desired result now follows from (1), (2), and (3).

Also solved by MICHEL BATAILLE, Rouen, France; WALTHER JANOUS, Ursulinen-gymnasium, Innsbruck, Austria; and the proposer.

Both Curtis and Janous pointed out that this problem is not new. Curtis cited the book The Red Book of Mathematical Problems *by K.S. Williams and K. Hardy, Dover, 1996; and Janous gave the reference* Mathematical Constants *by Steven R. Finch, Cambridge University Press, 2003.*

Alt obtained the following identity as a by-product:

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} \;=\; 2\sum_{n=1}^{\infty} \frac{1}{n^3}\,,$$

where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

The proposer gave the following comments: if we let

$$P(k) \;=\; \sum_{i_1=1}^{\infty} \cdots \sum_{i_k=1}^{\infty} \frac{1}{i_1 i_2 \cdots i_k (i_1 + i_2 + \cdots + i_k)}\,,$$

for $k = 1, 2, 3, \ldots$, *then clearly,* $P(1) = \zeta(2)$, *and the current problem shows that* $P(2) = 2\zeta(3)$, *where* $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ *denotes the Riemann Zeta function. He offered the conjecture that* $\zeta(k) = k\zeta(k+1)$ *for all* $k \in \mathbb{N}$. [Ed: *Here,* $P(1)$ *is interpreted to be* $\sum_{i_1=1}^{\infty} \frac{1}{i_1^2}$.]

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